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Dressing methods for geometric nets: II. Orthogonal and Egorov nets

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Abstract. The Grassmannian formalism of KP hierarchies is used to study geometric nets of orthogonal type and their subclass of Egorov nets. Efficient dressing methods for Cauchy propagators are provided which lead to wide families of explicit nets. Frobenius manifolds and solutions to the Witten–Dijkgraff–Verlinde–Verlinde associativity equations are also constructed.

(Some figures in this article are in colour only in the electronic version; see www.iop.org)

1. Introduction

In previous work [14] we have used the KP theory of integrable systems to derive efficient methods of solution for conjugate nets. The aim of this paper is to apply these methods to study orthogonal nets and their important subclass of Egorov nets.

The problem of constructing orthogonal nets [1], or flat diagonal metrics:

$$\mathrm{d}s^2 = \sum_{i=1}^{M} H_i^2 (\mathrm{d}u_i)^2$$

was one of the classical problems of differential geometry. The relevant underlying system of partial differential equations is [13, 18]

$$\frac{\partial \beta_{ij}}{\partial u_k} - \beta_{ik}\beta_{kj} = 0 \qquad i, j, k = 1, \dots, N \qquad \text{with} \quad i, j, k \text{ different}$$
$$\frac{\partial \beta_{ij}}{\partial u_i} + \frac{\partial \beta_{ji}}{\partial u_j} + \sum_{\substack{k=1,\dots,N \\ k\neq i,j}} \beta_{ki}\beta_{kj} = 0 \qquad i, j = 1, \dots, N \qquad i \neq j$$

where

$$\beta_{ij} := \frac{1}{H_i}\frac{\partial H_j}{\partial u_i} \qquad i \neq j.$$

Some years ago [4, 5, 8] it was found that the theory of orthogonal nets is closely related to the theory of integrable systems of hydrodynamic type in (1 + 1) dimensions. More recently [6], a particular type of orthogonal net (the ∂-invariant Egorov nets) defined by the conditions

$$\beta_{ij} = \beta_{ji} \tag{1}$$

and

$$\partial H_i = 0 \qquad \partial := \sum_j \frac{\partial}{\partial u_j} \tag{2}$$

appeared in the classification problem of massive topological field theories. A rich mathematical structure emerged in this context: the class of *Frobenius manifolds* [6–10]. Locally a Frobenius manifold is determined [6] by a flat metric

$$\mathrm{d}s^2 = \sum_{i,j=1}^{N} \eta^{ij} \mathrm{d}x_i \mathrm{d}x_j$$

and a commutative associative algebra structure

$$\partial_i \cdot \partial_j = \sum_k c_{ij}^k(x)\partial_k \qquad \partial_i := \frac{\partial}{\partial x^i} \qquad x^i := \sum_k \eta^{ik} x_k$$

with a unity ∂_1. The metric $\mathrm{d}s^2$ must be invariant with respect to this product, and *the deformed connection*

$$\nabla_i X^j := \partial_i X^j + z \sum_k c_{ik}^j(\boldsymbol{x}) X^k \tag{3}$$

where z is a spectral parameter, should have zero curvature.

In terms of the *data* $(\eta, c_{ij}^k(\boldsymbol{x}))$, the conditions for a Frobenius manifold can be formulated as [6]:

(i) η is a symmetric and non-degenerate matrix.
(ii) $c_{1j}^k = \delta_j^k$.
(iii) The coefficients $c_{ijk} := \sum_l \eta_{kl} c_{ij}^l$ are fully symmetric.
(iv) The linear system

$$\partial_i \xi_j = z \sum_k c_{ij}^k \xi_k \tag{4}$$

is compatible.

Basic objects in the theory of Frobenius manifolds are the systems of *deformed flat coordinates* $\theta_k(z, \boldsymbol{x})$ [6,7,9,10] for the connection (3). They are characterized by the conditions

$$\nabla_i \nabla_j \theta_k = 0 \tag{5}$$

which in turn are closely connected with the linear system (4). Indeed, equation (5) is equivalent to

$$\partial_i \partial_j \theta_k = z \sum_l c_{ij}^l(\boldsymbol{x}) \partial_l \theta_k \tag{6}$$

and therefore $\xi_j := \partial_j \theta_k$ verifies (4).

On the other hand, from the assumptions on $(\eta, c_{ij}^k(\boldsymbol{x}))$ one proves that there exists a function $F = F(\boldsymbol{x})$ (the *free energy function*) such that

$$c_{ijk} = \partial_i \partial_j \partial_k F$$

and, as a consequence of the associativity property of the algebra, the Witten–Dijkgraff–Verlinde–Verlinde (WDVV) equations [11,17] for F follow:

$$\sum_{r,s} \partial_i \partial_j \partial_r F \eta^{rs} \partial_s \partial_m \partial_k F = \sum_{r,s} \partial_i \partial_m \partial_s F \eta^{sr} \partial_r \partial_j \partial_k F. \tag{7}$$

Given a system of deformed flat coordinates normalized according to

$$\theta_i(0, \boldsymbol{x}) = x_i \qquad i = 1, \ldots, N \tag{8}$$

then (6) implies [6] that a free energy function can be derived from

$$\partial_i F(\boldsymbol{x}) = \frac{\partial \theta_i}{\partial z}(0, \boldsymbol{x}). \tag{9}$$

Furthermore, the coefficients of the expansions

$$\theta_i(z, \boldsymbol{x}) = \sum_{p \geqslant 0} h_{i,p}(\boldsymbol{x}) z^p \tag{10}$$

determine an infinite family of functionals:

$$H_{i,p}[\boldsymbol{x}] := \int h_{i,p+1}(\boldsymbol{x})\, \mathrm{d}t$$

which are in involution with respect to the Poisson bracket

$$\{x^i(t_1), x^j(t_2)\} := \eta^{ij} \delta'(t_1 - t_2).$$

The corresponding Hamiltonian systems constitute an integrable hierarchy of *systems of hydrodynamic type*.

Several methods from the theory of integrable systems have been proposed to generate Egorov nets, Frobenius manifolds and solutions to the WDVV equations (see, for instance, [12, 16]). In this paper we are concerned with the KP theory of conjugate nets from the point of view of the Grassmannian formalism. This scheme is explained in section 2, where the orthogonal and Egorov reductions are formulated in terms of simple conditions on the elements of the Grassmannian and the relationship with the theory of Frobenius manifolds is described. As we announced in [14], one of the main results of our analysis is that a Cauchy propagator

$$\frac{\partial \Psi}{\partial \bar{z}}(z, z', \boldsymbol{u}) = \pi \delta(z - z') \tag{11}$$

satisfying appropriate boundary conditions in the Grassmannian, is directly connected with basic geometric objects in the theory of orthogonal nets. In particular, this propagator is shown to provide systems of deformed flat coordinates for massive Frobenius manifolds. In section 3 dressing methods (which are the spectral generalizations of the Ribaucour transformations) for Cauchy propagators of the orthogonal and Egorov reductions are given. Finally, we construct and characterize several classes of explicit orthogonal and Egorov nets.

We must notice that in [3] an alternative $\bar{\partial}$ approach to the geometrical transformations of conjugate nets and quadrilateral lattices was given. It should also be stressed that a detailed study of the Cauchy propagator for quadrilateral lattices and a general $\bar{\partial}$ reduction theory which includes, as distinguished examples, the continuous and discrete orthogonal, symmetric, d-invariant and Egorov cases and the construction scheme for the separable solutions of the above geometric objects can be found in [2].

2. KP theory of geometric nets and Frobenius manifolds

2.1. Grassmannians and conjugate nets

The KP formalism of conjugate nets can be conveniently formulated in terms of the two families $\mathrm{Gr}_{\gamma(r)}$ and $\mathrm{Gr}^*_{\gamma(r)}$ of infinite-dimensional Grassmannians ($\gamma(r) := \{z \in \mathbb{C} : |z| = r\}$) introduced in [14]. The elements $W \in \mathrm{Gr}_{\gamma(r)}$ and $V \in \mathrm{Gr}^*_{\gamma(r)}$ are subsets of the space $H_{\gamma(r)}$ of Laurent series:

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

with coefficients a_n in the algebra $M_N(\mathbb{C})$ of $N \times N$ complex matrices, which converge on the circle $\gamma(r)$ and such that the projection on $H^+_{\gamma(r)}$:

$$P_+ : \sum_{n=-\infty}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} a_n z^n$$

is a bijective map. Furthermore, W and V are assumed to be left and right modules, respectively, for the algebra $M_N(\mathbb{C})$.

Each $W \in \mathrm{Gr}_{\gamma(r)}$ has an associated $W^* \in \mathrm{Gr}^*_{\gamma(r)}$ defined as the set of those $v \in H_{\gamma(r)}$ verifying

$$\int_{\gamma(r)} w(z)v(z)\,\mathrm{d}z = 0 \qquad \forall w \in W. \tag{12}$$

Given $W \in \mathrm{Gr}_{\gamma(r)}$ and $V \in \mathrm{Gr}^*_{\gamma(r)}$ the action of the KP flows are implemented by the multiplication operators

$$W(\boldsymbol{u}) = W\psi_0^{-1}(z, \boldsymbol{u}) \qquad V(\boldsymbol{u}) = \psi_0(z, \boldsymbol{u})V$$

where $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_N)$ denotes N infinite sequences

$$\boldsymbol{u}_i = (u_{i,1}, u_{i,2}, \ldots) \in \mathbb{C}^{N\cdot\infty}$$

and

$$\psi_0(z, \boldsymbol{u}) := \exp(\xi(z, \boldsymbol{u})) \qquad \xi(z, \boldsymbol{u}) := \sum_{n\geqslant 1} z^n \left(\sum_{i=1}^{N} u_{i,n} E_i\right) \qquad (E_i)_{jk} = \delta_{ij}\delta_{ik}.$$

An important notion in the theory of KP hierarchies is the concept of *normalization*. If $w = w(z, \boldsymbol{u})$ is such that either $w(\cdot, \boldsymbol{u}) \in W(\boldsymbol{u})$ or $w(\cdot, \boldsymbol{u}) \in W^*(\boldsymbol{u})$ for all appropriate $\boldsymbol{u}$ [14], then its normalization is defined by

$$\mathfrak{N}[w(z, \boldsymbol{u})] := P_+ w(z, \boldsymbol{u}).$$

It follows that functions $w(\cdot, \boldsymbol{u}) \in W(\boldsymbol{u})$ or $w(\cdot, \boldsymbol{u}) \in W^*(\boldsymbol{u})$ are uniquely determined by their normalization.

Given $W \in \mathrm{Gr}_{\gamma(r)}$ its associated KP wavefunction (the Baker–Akhiezer function) is defined as the unique function $\psi = \psi(z, \boldsymbol{u})$, such that its restriction to $\gamma(r)$ is the element of W which admits a convergent expansion of the form

$$\psi = \chi(z, \boldsymbol{u})\psi_0(z, \boldsymbol{u}) \qquad \chi(z, \boldsymbol{u}) = I_N + \sum_{n\geqslant 1} \frac{a_n(\boldsymbol{u})}{z^n}. \tag{13}$$

Similarly, the adjoint KP wavefunction associated with W is defined as the unique function $\psi^* = \psi^*(z, \boldsymbol{u})$, such that its restriction to $\gamma(r)$ is the element of W^* with a convergent expansion

$$\psi^* = \psi_0(z, \boldsymbol{u})^{-1}\chi^*(z, \boldsymbol{u}) \qquad \chi^*(z, \boldsymbol{u}) = I_N + \sum_{n\geqslant 1} \frac{a_n^*(\boldsymbol{u})}{z^n}. \tag{14}$$

The wavefunctions satisfy [14]

$$\frac{\partial \psi_i}{\partial u_k} = \beta_{ik}\psi_k \qquad \frac{\partial \psi_i^*}{\partial u_k} = \psi_k^*\beta_{ki} \qquad i \neq k \qquad u_k := u_{k,1} \tag{15}$$

where $\beta = \beta(\boldsymbol{u})$ is

$$\beta := a_1 = -a_1^* \tag{16}$$

and

$$\psi_i := (\psi_{i1}, \ldots, \psi_{iN}) \qquad \psi_i^* := \begin{pmatrix} \psi_{1i}^* \\ \vdots \\ \psi_{Ni}^* \end{pmatrix}.$$

The compatibility of these linear systems implies the Darboux equations:

$$\frac{\partial \beta_{ik}}{\partial u_k} = \beta_{ik}\beta_{kj} \qquad i, j \text{ and } k \text{ different.}$$

Furthermore, from (15) it follows that β_{ik} are the rotation coefficients for the family of conjugate nets with tangent vectors and Lamé coefficients given by $(\boldsymbol{X}_i)_j = X_{ij}$ and the rows $H_i = H_{li}$, $(l = 1, \ldots, N)$, respectively. Here

$$X(\boldsymbol{u}) := \int_{\mathbb{C}} \psi(z, \boldsymbol{u})\mathcal{N}(z)\,\mathrm{d}^2 z$$

$$H(\boldsymbol{u}) := \int_{\mathbb{C}} \mathcal{M}(z)\psi^*(z, \boldsymbol{u})\,\mathrm{d}^2 z$$

where $\mathcal{N}(z)$ and $\mathcal{M}(z)$ are appropriate matrix distributions. The corresponding conjugate nets $\boldsymbol{x}$ are the rows $x_i = \boldsymbol{x}_{li}$, $(l = 1, \ldots, N)$ of [14]

$$\boldsymbol{x}(\boldsymbol{u}) := \int_{\mathbb{C}\times\mathbb{C}} \mathcal{M}(z')\Psi(z, z')\mathcal{N}(z)\,\mathrm{d}^2 z\,\mathrm{d}^2 z' + \boldsymbol{x}_0. \tag{17}$$

Here, $\Psi(z, z')$ is the Cauchy propagator [14]

$$\Psi(z, z', \boldsymbol{u}) = \begin{cases} -\dfrac{1}{z'}\psi^*(z', \boldsymbol{u})\psi(z, \boldsymbol{u} + [z']) & \text{for} \quad |z| \leqslant |z'| \\ \dfrac{1}{z}\psi^*(z', \boldsymbol{u} - [z])\psi(z, \boldsymbol{u}) & \text{for} \quad |z'| \leqslant |z| \end{cases} \tag{18}$$

$$[z] := ([z]_1, \ldots, [z]_N) \qquad [z]_i := \left(\frac{1}{z}, \ldots, \frac{1}{nz^n}, \ldots\right).$$

It is a Green function for the $\overline{\partial}$ operator:

$$\frac{\partial \Psi}{\partial \bar{z}}(z, z', \boldsymbol{u}) = \pi\delta(z - z') \tag{19}$$

outside the disc $D(r) := \{z \in C : |z| < r\}$, and satisfies the following boundary conditions:

(1) The restriction of Ψ to $\gamma(r)$, as a function of z, is an element of W.
(2) As $z \to \infty$

$$\Psi(z, z', \boldsymbol{u}) = \mathcal{O}\left(\frac{1}{z}\right)\psi_0(z, \boldsymbol{u}).$$

The fundamental relation (17) is a consequence of the following differential equation [14]:

$$\frac{\partial \Psi}{\partial u_i}(z, z', \boldsymbol{u}) = \psi^*(z', \boldsymbol{u})E_i\psi(z, \boldsymbol{u}). \tag{20}$$

2.2. *Orthogonal reduction*

Definition 1. *An element $W \in \mathrm{Gr}_{\gamma(r)}$ satisfies the orthogonal reduction if, for every $v \in W^*$, it follows that $\tilde{v}(z) := zv(-z)^{\mathrm{t}}$ is an element of W.*

To analyse the consequences of this reduction we introduce the following involution in the space of KP parameters:

$$e(\boldsymbol{u}) = (e(\boldsymbol{u})_1, e(\boldsymbol{u})_2, \ldots, e(\boldsymbol{u})_N) \qquad e(\boldsymbol{u})_{i,n} = (-1)^{n+1}u_{i,n}.$$

Theorem 1. *If $W \in \mathrm{Gr}_{\gamma(r)}$ satisfies the orthogonal reduction, then:*

(i) The wavefunction and the adjoint wavefunction satisfy

$$z\psi^{*\mathrm{t}}(-z, e(\boldsymbol{u})) = \partial\psi(z, \boldsymbol{u}) + (\beta^{\mathrm{t}}(e(\boldsymbol{u})) - \beta(\boldsymbol{u}))\psi(z, \boldsymbol{u}) \tag{21}$$

where $\partial := \sum_i \frac{\partial}{\partial u_i}$.

(ii) The following identity for the Cauchy propagator holds:

$$z'\Psi(z, z', \boldsymbol{u}) - z\Psi^{\mathrm{t}}(-z', -z, e(\boldsymbol{u})) = -\psi^{\mathrm{t}}(-z', e(\boldsymbol{u}))\psi(z, \boldsymbol{u}). \tag{22}$$

(iii) The wavefunction satisfies

$$\psi^{\mathrm{t}}(-z, e(\boldsymbol{u}))\psi(z, \boldsymbol{u}) = I_N. \tag{23}$$

Proof. From the orthogonal reduction it is clear that $z\psi^{*\mathrm{t}}(-z, e(\boldsymbol{u}))$ belongs to W. Now, if we take into account that

$$\psi_0^{-1}(-z, e(\boldsymbol{u})) = \psi_0(z, \boldsymbol{u})$$

we find

$$z\psi^{*\mathrm{t}}(-z, e(\boldsymbol{u})) = z\left(1 + \frac{\beta^{\mathrm{t}}(e(\boldsymbol{u}))}{z} + \mathcal{O}\left(\frac{1}{z^2}\right)\right)\psi_0(z, \boldsymbol{u}) \qquad |z| = r$$

so that it has the same normalization as

$$\partial\psi(z, \boldsymbol{u}) + (\beta^{\mathrm{t}}(e(\boldsymbol{u})) - \beta(\boldsymbol{u}))\psi(z, \boldsymbol{u}).$$

Hence, statement (1) follows.

As for (2), let us denote

$$\phi(z, z', \boldsymbol{u}) := \Psi(z, z', \boldsymbol{u}) - \frac{z}{z'}\Psi^{\mathrm{t}}(-z', -z, e(\boldsymbol{u})). \tag{24}$$

From (18) we have that

$$\begin{aligned}\phi(z, z', \boldsymbol{u}) = &-\frac{1}{z'}\psi^*(z', \boldsymbol{u})\psi(z, \boldsymbol{u} + [z'])\\ &+\frac{z}{z'^2}\psi^{\mathrm{t}}(-z', e(\boldsymbol{u}))\psi^{*\mathrm{t}}(-z, e(\boldsymbol{u}) - [-z']) \qquad |z| \leqslant |z'|.\end{aligned}$$

As a consequence of the orthogonal reduction, the right-hand side of this identity, as a function of z, belongs to W. Moreover, it can be analytically extended outside $D(r)$ with only one possible singularity at $z = z'$. But from (24) it is easy to conclude that this singularity is avoidable. Therefore, the Laurent expansion of $\phi(z, z', \boldsymbol{u})$ as $z \to \infty$ can be extended to $\gamma(r)$. Thus, by using (18) for $|z| \geqslant |z'|$, one gets

$$\phi(z, z', \boldsymbol{u}) = \left(-\frac{1}{z'}\psi^{\mathrm{t}}(-z', e(\boldsymbol{u})) + \mathcal{O}\left(\frac{1}{z}\right)\right)\psi_0(z, \boldsymbol{u}) \qquad |z| = r.$$

This implies that

$$\phi(z, z', \boldsymbol{u}) = -\frac{1}{z'}\psi^{\mathrm{t}}(-z', e(\boldsymbol{u}))\psi(z, \boldsymbol{u})$$

which proves statement (2).

Finally, (3) follows by letting $z' \to z$ in (22). □

The next result establishes the relationship between the theory of orthogonal nets and the formalism of KP hierarchies. The following notational convention is used:

$$\mathfrak{u} = (\mathfrak{u}_1, \ldots, \mathfrak{u}_N) \in \mathbb{C}^{N\cdot\infty} \qquad \begin{cases} \mathfrak{u}_{i,n} = 0 & \text{for} \quad n \ \text{even} \\ \mathfrak{u}_{i,n} = u_{i,n} & \text{for} \quad n \ \text{odd.} \end{cases}$$

Theorem 2. *Let W be an element of $\mathrm{Gr}_{\gamma(r)}$ which satisfies the orthogonal reduction and $H_i = H_{li}$, $(l = 1, \ldots, N)$ be a row of the matrix function*

$$H(\mathrm{u}) := \int_{\mathbb{C}} \mathcal{M}(z)\psi^*(z, \mathrm{u})\,\mathrm{d}^2 z. \tag{25}$$

Then the diagonal metric

$$\mathrm{d}s^2 = \sum_{i=1}^{N} H_i^2 (\mathrm{d}u_i)^2 \tag{26}$$

is flat. Moreover, for any pair of matrices $\mathcal{N}$ ($\det\mathcal{N} \neq 0$) and $\boldsymbol{x}_0$ the corresponding rows $x_i = \boldsymbol{x}_{li}$, $(l = 1, \ldots, N)$ of

$$\boldsymbol{x}(\mathrm{u}) := \int_{\mathbb{C}} \mathcal{M}(z')\Psi(0, z', \mathrm{u})\mathcal{N}\,\mathrm{d}^2 z' + \boldsymbol{x}_0 \tag{27}$$

determine flat coordinate systems with

$$\mathrm{d}s^2 = \sum_{i,j=1}^{N} \eta^{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j \qquad \eta := (\mathcal{N}^{\mathrm{t}}\mathcal{N})^{-1}. \tag{28}$$

Proof. According to the above discussion we have

$$\frac{\partial x_i}{\partial u_j} = H_j(\boldsymbol{X}_j)_i$$

where

$$(\boldsymbol{X}_i)_j = (\psi(0)\mathcal{N})_{ij}$$

so that

$$\frac{\partial x_i}{\partial u_j} = H_j(\psi(0)\mathcal{N})_{ji}.$$

By assuming that the Lamé coefficients do not vanish, we have

$$\frac{\partial(x_1, \ldots, x_N)}{\partial(u_1, \ldots, u_N)} \neq 0.$$

Thus, there exist local functions $u_i = u_i(x_1, \ldots, x_N)$. Hence, by writing (26) in terms of the coordinate system $\{x_i\}$ it follows that

$$\mathrm{d}s^2 = \sum_{j,j'=1}^{N} \eta^{jj'}\,\mathrm{d}x_j\,\mathrm{d}x_{j'} \qquad \eta^{jj'} := \sum_{i=1}^{N} H_i^2 \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_{j'}}.$$

Furthermore, one has

$$\sum_{j,j'=1}^{N} \frac{1}{H_k}\frac{\partial x_j}{\partial u_k}\eta^{jj'}\frac{1}{H_{k'}}\frac{\partial x_{j'}}{\partial u_{k'}} = \delta_{kk'}$$

or in matrix form

$$\psi(0, \mathrm{u})\mathcal{N}\eta(\mathrm{u})\mathcal{N}^{\mathrm{t}}\psi(0, \mathrm{u})^{\mathrm{t}} = I_N.$$

Therefore, by using the identity (23), we get (28). □

Comments. Observe that (27) corresponds to $\mathcal{N}(z) = \mathcal{N}\delta(z)$ in (17), and that the rows of $\boldsymbol{x}$ describe a set of parallel orthogonal nets provided $\mathcal{N}$ is an orthogonal matrix.

2.3. Egorov reduction

The Egorov reduction is an special case of the orthogonal reduction. It can be defined as follows.

Definition 2. *An element $W \in \mathrm{Gr}_{\gamma(r)}$ satisfies* the Egorov reduction *if*

(i) For every $w \in W$ the function $\tilde{w}(z) := zw(z)$ is also in W.
(ii) For every $v \in W^*$ the function $\tilde{v}(z) := v(-z)^{\mathrm{t}}$ is in W.

One can show that an equivalent characterization is as given in the following proposition.

Proposition 1. *An element $W \in \mathrm{Gr}_{\gamma(r)}$ satisfies* the Egorov reduction *if and only if:*

(i) W satisfies the orthogonal reduction.
(ii) For every $v \in W^*$ the function $\tilde{v}(z) := v(-z)^{\mathrm{t}}$ is in W.

The Egorov reduction implies the following properties for the Baker functions and the Cauchy propagators.

Theorem 3. *If $W \in \mathrm{Gr}_{\gamma(r)}$ satisfies the Egorov reduction, then:*

(i) The wavefunction and the adjoint wavefunction satisfy

$$\psi^{*\mathrm{t}}(-z, e(\boldsymbol{u})) = \psi(z, \boldsymbol{u}) \qquad \partial\psi(z, \boldsymbol{u}) = z\psi(z, \boldsymbol{u}). \tag{29}$$

(ii) The Cauchy propagator is given by

$$\Psi(z, z', \boldsymbol{u}) = \frac{\psi^{\mathrm{t}}(-z', e(\boldsymbol{u}))\psi(z, \boldsymbol{u})}{z - z'}. \tag{30}$$

Proof. We have

$$\psi^{*\mathrm{t}}(-z, e(\boldsymbol{u})) = \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right)\psi_0(z, \boldsymbol{u}) \qquad |z| = r.$$

Moreover, as W verifies the Egorov reduction, $\psi^{*\mathrm{t}}(-z, e(\boldsymbol{u}))$ belongs to W. Hence, as it has the same normalization as $\psi(z, \boldsymbol{u})$, the first identity of (29) follows. Similarly, the Egorov reduction implies that $z\psi(z, \boldsymbol{u})$ belongs to W and, due to the fact that this function has the same normalization as $\partial\psi(z, \boldsymbol{u})$, the second identity in (29) follows.

To prove (30) we observe that, from (18), (29) and taking into account that

$$e([z]) = -[-z]$$

we get

$$\Psi^{\mathrm{t}}(-z', -z, e(\boldsymbol{u})) = \Psi(z, z', \boldsymbol{u})$$

then (22) leads to (30). □

Theorem 4. *Let W be an element of $\mathrm{Gr}_{\gamma(r)}$ which satisfies the Egorov reduction and $H_i = H_{li}$, $(l = 1, \ldots, N)$ be a row of the matrix function*

$$H(\mathrm{u}) := \int_{\mathbb{C}} \mathcal{M}(z)\psi^*(z, \mathrm{u})\, \mathrm{d}^2 z.$$

Then, the rotation coefficients *of the flat diagonal metric*

$$\mathrm{d}s^2 = \sum_{i=1}^{N} H_i^2 (\mathrm{d}u_i)^2 \tag{31}$$

satisfy the symmetry condition

$$\beta_{ij}(\mathrm{u}) = \beta_{ji}(\mathrm{u}). \tag{32}$$

Proof. This result is just a consequence of (16) and (29). □

We observe that, by using (29) in the above theorem, one has

$$H_{li}^2(\mathfrak{u}) = \int_{\mathbb{C}\times\mathbb{C}} \sum_{j,j'=1}^{N} \mathcal{M}_{lj}(z')\psi_{ji}^*(z',\mathfrak{u})\psi_{ij'}(z,\mathfrak{u})\mathcal{M}_{lj'}(-z)\,\mathrm{d}^2z\,\mathrm{d}^2z'$$

and so by recalling (20) we may write

$$H_{li}^2(\mathfrak{u}) = \int_{\mathbb{C}\times\mathbb{C}} \sum_{j,j'=1}^{N} \mathcal{M}_{lj}(z')\frac{\partial\Psi_{jj'}(z,z',\mathfrak{u})}{\partial u_i}\mathcal{M}_{lj'}(-z)\,\mathrm{d}^2z\,\mathrm{d}^2z'.$$

Therefore, the potential Θ of the corresponding Egorov metric

$$\mathrm{d}s^2 = \sum_{i=1}^{N}\frac{\partial\Theta}{\partial u_i}(\mathrm{d}u_i)^2$$

is

$$\Theta = \left(\int_{\mathbb{C}\times\mathbb{C}} \mathcal{M}(z')\Psi(z,z')\mathcal{M}^{\mathrm{t}}(-z)\,\mathrm{d}^2z\,\mathrm{d}^2z'\right)_{ll}.$$

Theorem 5. *Let W be an element of $\mathrm{Gr}_{\gamma(r)}$ which satisfies the Egorov reduction and $H_i = H_{li}$, $(l = 1,\ldots,N)$ be a row of the matrix function*

$$H(\mathfrak{u}) := \mathcal{M}\psi^*(0,\mathfrak{u}). \tag{33}$$

Then, the metric

$$\mathrm{d}s^2 = \sum_{i=1}^{N} H_i^2(\mathrm{d}u_i)^2$$

is a ∂-invariant Egorov metric. Furthermore, for any non-singular matrix $\mathcal{N}$ the corresponding rows $x_i = \boldsymbol{x}_{li}$, $(l = 1,\ldots,N)$ of

$$\boldsymbol{x}(\mathfrak{u}) := \lim_{z\to0}\mathcal{M}\left(\Psi(z,0,\mathfrak{u}) - \frac{1}{z}\right)\mathcal{N} \tag{34}$$

determine flat coordinate systems:

$$\mathrm{d}s^2 = \sum_{i,j=1}^{N}\eta^{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j \qquad \eta = (\mathcal{N}^{\mathrm{t}}\mathcal{N})^{-1}. \tag{35}$$

Proof. Notice that, according to the Egorov reduction

$$H(\mathfrak{u}) = \mathcal{M}\psi^*(0,\mathfrak{u}) = \mathcal{M}\psi^{\mathrm{t}}(0,\mathfrak{u})$$

so that from (29) one finds $\partial H_i = 0$. The rest of the proof follows by observing that (33) is obtained from (25) by setting $\mathcal{M}(z) = \mathcal{M}\delta(z)$. Thus, (34) is obtained by regularizing the corresponding expression for $\boldsymbol{x}(\mathfrak{u})$ in (27). □

2.4. Frobenius manifolds and WDVV equations

Our next aim now is to show the relationship between the theory of integrable systems of KP type and the theory of Frobenius manifolds [6, 7, 9, 10].

Theorem 6. *Let W be an element of $\mathrm{Gr}_{\gamma(r)}$ which satisfies the Egorov reduction. Then, for any non-singular matrix $\mathcal{N}$ the functions*

$$\theta_i(z, \mathrm{u}) := \left(\mathcal{N}^{\mathrm{t}}\left(\Psi(z, 0, \mathrm{u}) - \frac{1}{z}\right)\mathcal{N}\right)_{1i} \qquad i = 1, \ldots, N \tag{36}$$

are a system of normalized deformed flat coordinates for a Frobenius manifold determined by:

(1) The ∂-invariant Egorov metric:

$$\mathrm{d}s^2 = \sum_{i=1}^{N} H_i^2(\mathrm{d}u_i)^2 \qquad H_i(\mathrm{u}) := (\psi(0, \mathrm{u})\mathcal{N})_{i1}. \tag{37}$$

(2) The system of flat coordinates:

$$x_i := \theta_i(0, \mathrm{u}) \qquad i = 1, \ldots, N \tag{38}$$

$$\mathrm{d}s^2 = \sum_{i,j=1}^{N} \eta^{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j \qquad \eta = (\mathcal{N}^{\mathrm{t}}\mathcal{N})^{-1}. \tag{39}$$

(3) The structure constants:

$$c_{ij}^{l} = \sum_{k=1}^{N} \frac{\partial u_k}{\partial x^i}\frac{\partial u_k}{\partial x^j}\frac{\partial x^l}{\partial u_k}. \tag{40}$$

Proof. From theorem 5 it is clear that (37) is a ∂-invariant Egorov metric and that (38) defines a system of flat coordinates. By introducing the matrix functions

$$X := \psi(0, \mathrm{u})\mathcal{N} \qquad Y := \psi(z, \mathrm{u})\mathcal{N}$$

and by taking (30) into account, we may write

$$\theta_i(z, \mathrm{u}) = \frac{1}{z}(X^{\mathrm{t}}Y - \eta^{-1})_{1i}. \tag{41}$$

Moreover, (20) becomes

$$\frac{\partial}{\partial u_i}(X^{\mathrm{t}}Y) = z(X^{\mathrm{t}}E_iY) \tag{42}$$

so that

$$\frac{\partial \theta_j}{\partial u_i} = (X^{\mathrm{t}}E_iY)_{1j}. \tag{43}$$

On the other hand, from theorem 5 we have that

$$\frac{\partial x^j}{\partial u_i} = \sum_k H_i X_{ik}\eta^{kj} \qquad x^i := \sum_k \eta^{ik}x_k \tag{44}$$

and therefore

$$\frac{\partial u_i}{\partial x^j} = \sum_k \eta_{jk}(X^{-1})_{ki}H_i^{-1} = X_{ij}H_i^{-1} \tag{45}$$

where

$$\eta^{-1} = (\eta_{ij}) = \mathcal{N}^{\mathrm{t}}\mathcal{N} = X^{\mathrm{t}}X.$$

From (43)–(45) and by noticing that $H_i = X_{i1}$ one finds at once that

$$\frac{\partial \theta_i}{\partial x^j} = (X^{\mathrm{t}} Y)_{ji}.$$

Thus, (42) and (45) imply

$$\frac{\partial^2 \theta_i}{\partial x^j \partial x^k} = z \sum_l (X^{\mathrm{t}} E_l Y)_{ki} \frac{\partial u_l}{\partial x^j} = z \sum_l \frac{\partial u_l}{\partial x^j} X_{lk} Y_{li}.$$

Therefore, by observing that

$$\begin{aligned} Y_{ij} &= (X \eta X^{\mathrm{t}} Y)_{ij} \\ &= \sum_k (X\eta)_{ik} (X^{\mathrm{t}} Y)_{kj} \end{aligned}$$

and using (44) and (45) we get

$$\frac{\partial^2 \theta_i}{\partial x^j \partial x^k} = z \sum_{l,m} \frac{\partial u_l}{\partial x^j} \frac{\partial u_l}{\partial x^k} \frac{\partial x^m}{\partial u_l} \frac{\partial \theta_i}{\partial x^m}$$

which shows (6) and (40). Notice also that from (38) the normalization condition (8) is satisfied.

The rest of the proof follows by observing that

$$c_{ijk} := \sum_l \eta_{il} c^l_{jk} = \sum_l H_l^{-1} X_{lj} X_{lk} X_{li}$$

is fully symmetric. Furthermore

$$\frac{\partial u_i}{\partial x^1} = X_{i1} H_i^{-1} = 1$$

so that

$$c^j_{1i} = \delta^j_i.$$

□

Notice that, as a consequence of (36) and (10), every $W \in \mathrm{Gr}_{\gamma(r)}$ which satisfies the Egorov reduction determines a hierarchy of systems of hydrodynamic type with Hamiltonian densities given by

$$h_{i,p}(\boldsymbol{x}) = \frac{1}{(p+1)!} \frac{\partial^{p+1}}{\partial z^{p+1}} (\mathcal{N}^{\mathrm{t}} z \Psi(z, 0, \mathrm{u}) \mathcal{N})_{1i}|_{z=0}.$$

3. Solution methods

3.1. Dressing conjugate nets

We first describe in brief the dressing method presented in [14]. Let $D(r)$ and $D(\tilde{r})$ be two discs centred at the origin with $r < \tilde{r}$. Denote by $\gamma(r)$ and $\gamma(\tilde{r})$ their respective boundaries, and by A the annulus $D(\tilde{r}) - D(r)$.

Definition 3. *Given a matrix distribution $R = R(z, z')$ with support in $A \times A$, it determines a 'dressing transformation':*

$$T_R : \mathrm{Gr}_{\gamma(r)} \mapsto \mathrm{Gr}_{\gamma(\tilde{r})} \qquad W \mapsto \tilde{W} \tag{46}$$

where for every $W \in \mathrm{Gr}_{\gamma(r)}$ the corresponding $\tilde{W} \in \mathrm{Gr}_{\gamma(\tilde{r})}$ is the set of boundary values on $\gamma(\tilde{r})$ of matrix functions $w = w(z)$ satisfying the $\bar{\partial}$ equation

$$\frac{\partial w}{\partial \bar{z}}(z) = \int_A w(z') R(z', z)\, \mathrm{d}^2 z' \qquad z \in A$$

and such that the restriction of w to $\gamma(r)$ is an element of W.

It was proved in [14] that the Cauchy propagators Ψ and $\tilde{\Psi}$ associated with W and $\tilde{W}$, respectively, are related by

$$\tilde{\Psi}(z, z') = \Psi(z, z') + \int_A c(z', z'')\Psi(z, z'')\, \mathrm{d}^2 z'' \tag{47}$$

where $c(z, z')$ is the solution of the integral equation

$$c(z', z) = \frac{1}{\pi}\int_A \Psi(z'', z')R(z'', z)\, \mathrm{d}^2 z'' + \frac{1}{\pi}\int_{A\times A} c(z', z''')\Psi(z'', z''')R(z'', z)\, \mathrm{d}^2 z''\, \mathrm{d}^2 z'''. \tag{48}$$

For separable kernels

$$R(z, z') = \pi \sum_{k=1}^{m}\sum_{l=1}^{n} C_{k\ell} f_k(z) g_\ell(z'). \tag{49}$$

Equation (48) can be solved explicitly. Here $C_{k\ell}$ are $N \times N$ constant complex matrices, and f_k, g_ℓ are scalar distributions. To describe the corresponding solution let us introduce the following notation:

$$\begin{aligned}
\mu_k(z) &:= \int_A \Psi(z', z) f_k(z')\, \mathrm{d}^2 z' \qquad k = 1, \ldots, m\\
\nu_\ell(z) &:= \int_A \Psi(z, z') g_\ell(z')\, \mathrm{d}^2 z' \qquad \ell = 1, \ldots, n\\
\omega_{\ell k} &:= \int_{A\times A} \Psi(z', z'') f_k(z') g_\ell(z'')\, \mathrm{d}^2 z'\, \mathrm{d}^2 z'' \qquad k = 1, \ldots, m \qquad \ell = 1, \ldots, n
\end{aligned} \tag{50}$$

and the matrices

$$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m) : A \to M_{N\times mN}(\mathbb{C}) \qquad \boldsymbol{\nu} = \begin{pmatrix}\nu_1\\ \vdots\\ \nu_n\end{pmatrix} : A \to M_{nN\times N}(\mathbb{C})$$

$$\boldsymbol{C} = (C_{kl}) \in M_{mN\times nN}(\mathbb{C}) \qquad \boldsymbol{\omega} = (\omega_{\ell k}) \in M_{nN\times mN}(\mathbb{C}).$$

Then, we have [14]

$$\tilde{\Psi}(z, z') = \Psi(z, z') + \boldsymbol{\mu}(z')\boldsymbol{C}(1 - \boldsymbol{\omega}\boldsymbol{C})^{-1}\boldsymbol{\nu}(z).$$

In [14] we also showed that:

(1) The dressing transformations for the Baker function $\psi(z)$, adjoint Baker functions $\psi^*(z)$ and the matrix of rotation coefficients β are

$$\begin{aligned}
\tilde{\psi}(z) &= \psi(z) + \boldsymbol{\varphi}\boldsymbol{C}(1 - \boldsymbol{\omega}\boldsymbol{C})^{-1}\boldsymbol{\nu}(z)\\
\tilde{\psi}^*(z) &= \psi^*(z) + \boldsymbol{\mu}(z)\boldsymbol{C}(1 - \boldsymbol{\omega}\boldsymbol{C})^{-1}\boldsymbol{\varphi}^*\\
\tilde{\beta} &= \beta + \boldsymbol{\varphi}\boldsymbol{C}(1 - \boldsymbol{\omega}\boldsymbol{C})^{-1}\boldsymbol{\varphi}^*
\end{aligned}$$

with

$$\boldsymbol{\varphi} := (\varphi_1, \ldots, \varphi_m) \qquad \varphi_k = \int_A \psi(z) f_k(z)\, \mathrm{d}^2 z$$

$$\boldsymbol{\varphi}^* := \begin{pmatrix}\varphi_1^*\\ \vdots\\ \varphi_n^*\end{pmatrix} \qquad \varphi_\ell^* = \int_A \psi^*(z) g_\ell(z)\, \mathrm{d}^2 z.$$

(2) Each $W \in \mathrm{Gr}_{\gamma(r)}$ determines a set of parallel conjugate nets given by the rows of

$$\boldsymbol{x} := \int_{\mathbb{C}^2} \mathcal{M}(z')\psi(z,z')\mathcal{N}(z)\,\mathrm{d}^2z\,\mathrm{d}^2z' + \boldsymbol{x}_0$$

where $\mathcal{M}(z)$ and $\mathcal{N}(z)$ $N \times N$ are complex matrix distributions and $\boldsymbol{x}_0$ is a constant $N \times N$ matrix. The dressed nets are then given by the corresponding rows of the matrix

$$\tilde{\boldsymbol{x}} := \boldsymbol{x} + \mathcal{M}\boldsymbol{C}(1 - \boldsymbol{\omega}\boldsymbol{C})^{-1}\mathcal{N}$$

where

$$\mathcal{M} = \int_{\mathbb{C}} \mathcal{M}(z)\boldsymbol{\mu}(z)\,\mathrm{d}^2z \qquad \mathcal{N} = \int_{\mathbb{C}} \boldsymbol{\nu}(z)\mathcal{N}(z)\,\mathrm{d}^2z.$$

3.2. Dressing orthogonal nets

According to the above analysis, a dressing transformation preserves the orthogonal reduction if it satisfies

$$v \in \tilde{W}^* \Rightarrow zv^{\mathrm{t}}(-z) \in \tilde{W}.$$

On the other hand, for $v \in \tilde{W}^*$ we have

$$\frac{\partial v}{\partial \bar{z}} = -\int_A R(z,z')v(z')\,\mathrm{d}^2z'$$

so that

$$\frac{\partial(zv^{\mathrm{t}}(-z))}{\partial \bar{z}} = \int_A (z'v^{\mathrm{t}}(-z'))\left[\frac{z}{z'}R^{\mathrm{t}}(-z,-z')\right]\mathrm{d}^2z'.$$

Thus, if the kernel satisfies the condition

$$zR(z,z') - z'R^{\mathrm{t}}(-z',-z) = 0 \tag{51}$$

then its corresponding dressing transformation preserves the orthogonal reduction. Examples of separable kernels of this type are

$$R(z,z') = -\pi z' \sum_{k,\ell=1}^{n} f_k(z)C_{k\ell}f_\ell(-z') \tag{52}$$

where $C_{k\ell} + C^{\mathrm{t}}_{\ell k} = 0$ and $\{f_k(z)\}_{k=1}^n$ are scalar distributions. These kinds of separable kernels are obtained from the general class considered in [14] by setting $m = n$ and $g_\ell(z) = -zf_\ell(-z)$.

Let us consider the simplest dressing, with $n = 1$ and only one spectral distribution, $f(z)$, of the Cartesian net corresponding to $H_1 = \cdots = H_N = 1$ and $\boldsymbol{X}_i = e_i$, $i = 1,\ldots,N$ (the canonical basis of $\mathbb{R}^N$). In this case the functions defining the dressing have the following form:

$$\mu_i(u_i) := \int_A \frac{\mathrm{e}^{zu_i}}{z} f(z)\,\mathrm{d}^2z \qquad \nu_i(u_i) := -\int_A \frac{\mathrm{e}^{-zu_i}}{z}(-zf(-z))\,\mathrm{d}^2z = \mu_i'(u_i)$$

$$\phi_i(u_i) := \int_A \mathrm{e}^{zu_i} f(z)\,\mathrm{d}^2z = \mu_i'(u_i) \qquad \phi_i^*(u_i) := \int_A \mathrm{e}^{-zu_i}(-zf(-z))\,\mathrm{d}^2z = \mu_i''(u_i)$$

$$\omega_i(u_i) := \int_{A\times A} \frac{\mathrm{e}^{(z-z')u_i}}{z-z'} f(z)(-z'f(-z'))\,\mathrm{d}^2z\,\mathrm{d}^2z' = \frac{1}{2}\mu_i'(u_i)^2.$$

Thus, the dressing formulae are given as follows.

Proposition 2. *The next data characterize an orthogonal net*

$$(\tilde{x}_1, \dots, \tilde{x}_N) = (u_1, \dots, u_N) + (\mu_1, \dots, \mu_N) C (I_N - \tfrac{1}{2}\operatorname{diag}((\mu_1')^2, \dots, (\mu_N')^2) C)^{-1} \operatorname{diag}(\mu_1', \dots, \mu_N')$$

$$\tilde{H}_j := 1 + (\mu_1, \dots, \mu_N) C (I_N - \tfrac{1}{2}\operatorname{diag}((\mu_1')^2, \dots, (\mu_N')^2) C)^{-1} \boldsymbol{e}_j^{\mathrm{t}} \mu_j''.$$

Comments.

- Observe that all the geometrical data are parametrized in terms of μ and its first and second derivatives.
- In this case one can readily prove
$$\Delta := \det(I_N - \tfrac{1}{2}\operatorname{diag}((\mu_1')^2, \dots, (\mu_N')^2) C) \neq 0$$
so that there are not singularities in the net. To see this we denote by $A_{i_1,\dots,i_k}$ ($i_1, \dots, i_k$ are different numbers in the sequence $1, \dots, N$) the matrix built up with the $i_1, \dots, i_k$, rows and columns of a $N \times N$ matrix A and recall that
$$\det(I_N + A) = \sum_{k=0}^{N} \det(A_{i_1,\dots,i_k}).$$
In our case $A = -\frac{1}{2}\operatorname{diag}((\mu_1')^2, \dots, (\mu_N')^2) C$, with C skew. Thus the odd order invariants cancel, and the final expression is
$$\Delta = 1 + \sum_{k=1}^{[N/2]} \sum_{i_1,\dots,i_{2k}} \frac{1}{4^k} \operatorname{Pf}(C_{i_1,\dots,i_{2k}})^2 (\mu_{i_1}')^2, \dots, (\mu_{i_{2k}}')^2$$
which is bigger than 1. Here we have used the Pfaffian of a skew matrix.
- An interesting aspect of orthogonal nets is that a given coordinate hypersurface is intersected by the others in curvature lines. Moreover, the principal curvatures can be computed easily. In fact, if we deal with the ith coordinate hypersurface $u_i = \text{constant}$, then the $(N-1)$ principal curvatures are given by
$$\kappa_j^{(i)} = -\frac{\beta_{ij}}{H_j} \qquad j \neq i.$$
Hence, from our simple dressing we obtain hypersurfaces parametrized by curvature lines and with principal curvatures given by
$$\kappa_j^{(i)} = -\frac{\mu_i' \mu_j'' \boldsymbol{e}_i C (I_N - \frac{1}{2}\operatorname{diag}((\mu_1')^2, \dots, (\mu_N')^2) C)^{-1} \boldsymbol{e}_j^{\mathrm{t}}}{1 + (\mu_1, \dots, \mu_N) C (I_N - \frac{1}{2}\operatorname{diag}((\mu_1')^2, \dots, (\mu_N')^2) C)^{-1} \boldsymbol{e}_j^{\mathrm{t}} \mu_j''} \qquad j \neq i.$$

3.2.1. Example: For $N = 2$,
$$\Delta = 1 + \frac{c_{12}^2}{4} (\mu_1')^2 (\mu_2')^2$$
and the net is
$$\tilde{x}_i = u_i + \frac{1}{\Delta}\left[-\frac{c_{12}^2}{2}\mu_i (\mu_j')^2 + \mu_j c_{ji}\right]\mu_i'$$
with i, j cyclic. The curvature of the ith ($i = 1, 2$) lines coordinates is given by
$$\kappa^{(i)} = -\frac{\beta_{ij}}{H_j} = -\frac{c_{ij}\mu_i'\mu_j''}{\Delta + [-\frac{c_{12}^2}{2}\mu_j(\mu_i')^2 + \mu_i c_{ij}]\mu_j''}.$$

We now present some explicit examples.

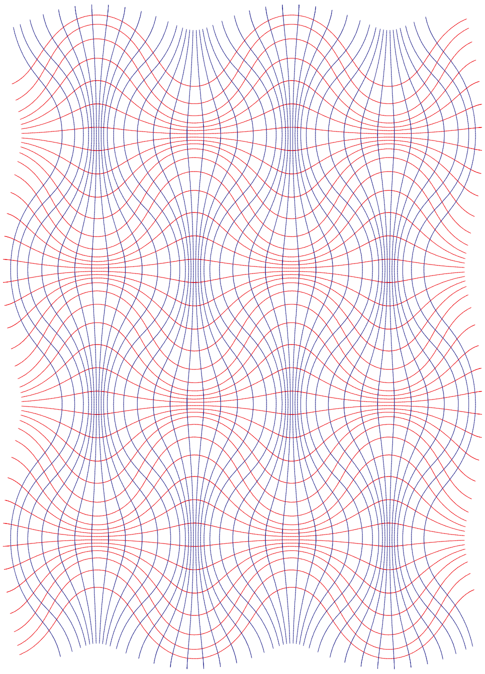


Figure 1.

Elliptic periodic nets. We take

$$\mu(u_1) := \tfrac{1}{2}\mathrm{sn}(2u_1|\tfrac{1}{2})$$

the elliptic sine of Jacobi with argument $m = 1/2$ (an argument which we omit in the following) and $c_{12} = 1$. Then, the net is given by

$$x_1(u_1, u_2) := u_1 - \mathrm{cn}2u_1\,\mathrm{dn}2u_1 \frac{2\mathrm{sn}2u_2 - \mathrm{cn}^2 2u_2\,\mathrm{dn}^2 2u_2\,\mathrm{sn}2u_1}{4 + \mathrm{cn}^2 2u_1\,\mathrm{dn}^2 2u_1\,\mathrm{cn}^2 2u_2\,\mathrm{dn}^2 2u_2}$$

$$x_2(u_1, u_2) := u_2 + \mathrm{cn}2u_2\,\mathrm{dn}2u_2 \frac{2\mathrm{sn}2u_1 - \mathrm{cn}^2 2u_1\,\mathrm{dn}^2 2u_1\,\mathrm{sn}2u_2}{4 + \mathrm{cn}^2 2u_1\,\mathrm{dn}^2 2u_1\,\mathrm{cn}^2 2u_2\,\mathrm{dn}^2 2u_2}.$$

One can check that $H_1 H_2 \neq 0$ and thus the periodic orthogonal net is nonsingular and locally regular. In figure 1 we plot the coordinate lines.

Comment. This elliptic net is just a particular example of the periodic orthogonal nets that can be constructed from a periodic function μ. Indeed, we could take the spectral measure to be a general Dirac comb of the form

$$f(z) = \sum_i A_i \delta(z - \mathrm{i}p_i).$$

In this elliptic case we have taken

$$f(z) = \frac{\pi^2}{2\sqrt{m}K^2} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 - q^{2n+1}} (2n+1) \left(\delta\left(z - \mathrm{i}(2n+1)\frac{\pi}{2K}\right) + \delta\left(z + \mathrm{i}(2n+1)\frac{\pi}{2K}\right)\right)$$

with K and K' being the real and imaginary quarter periods

$$K = \int_0^{\pi/2} (1 - m\sin^2\theta)^{-1/2}\,\mathrm{d}\theta \qquad K' = \int_0^{\pi/2} (1 - (1-m)\sin^2\theta)^{-1/2}\,\mathrm{d}\theta$$

and the corresponding nome $q = \exp(-\pi K'/K)$.

Hermite nets. In [14] the Hermite conjugate nets of (r, s) type were constructed. Moreover, it was shown that, for $s > 0$, the net describes a Gaussian localized deformation of the Cartesian net. Under the orthogonal reduction condition the corresponding Hermite spectral measures must be taken as

$$f(z) = \frac{Ak}{\sqrt{2\pi}}\delta\left(\frac{z+\bar{z}}{2}\right) z^r \mathrm{e}^{k^2 z^2/2}$$
$$g(z) = \frac{Ak}{\sqrt{2\pi}}(-1)^r\delta\left(\frac{z+\bar{z}}{2}\right) z^{r+1} \mathrm{e}^{k^2 z^2/2}$$

and then μ becomes

$$\mu(u) = \begin{cases} Ak\sqrt{\dfrac{\pi}{2}}\mathrm{erf}\left(\dfrac{u}{\sqrt{2}k}\right) & r = 0 \\ \dfrac{A}{(\sqrt{2}k)^{r-1}}H_{r-1}\left(-\dfrac{u}{\sqrt{2}k}\right)\mathrm{e}^{-\frac{u^2}{2k^2}} & r > 0 \end{cases}$$

where we are using the error function erf and the Hermite polynomials H_r. The only Hermite orthogonal nets are among the Hermite conjugate nets of $(r, r+1)$-type; hence they are Gaussian localized in the sense of [14].

In our example we take

$$\mu(u) = \tfrac{1}{2}\,\mathrm{erf}\, u$$

and $c_{12} = 1$. The net is given by

$$x_1(u_1, u_2) := u_1 - \sqrt{\pi}\mathrm{e}^{-u_1^2}\frac{2\pi\,\mathrm{erf}\, u_2 + \mathrm{e}^{-2u_2^2}\mathrm{erf}\, u_1}{4\pi^2 + \mathrm{e}^{-2(u_1^2+u_2^2)}}$$
$$x_2(u_1, u_2) := u_2 + \sqrt{\pi}\mathrm{e}^{-u_2^2}\frac{2\pi\,\mathrm{erf}\, u_1 - \mathrm{e}^{-2u_1^2}\mathrm{erf}\, u_2}{4\pi^2 + \mathrm{e}^{-2(u_1^2+u_2^2)}}.$$

The corresponding plot is shown in figure 2.

The Gaussian localization of the net, which is exhibited in the plot, can also be described by showing that the curvatures $\kappa_2^{(1)}$ and $\kappa_1^{(2)}$ of the coordinate lines are localized. For example, $\kappa_2^{(1)}$ is the curvature of the coordinate lines $u_1 = cte$ and, as we change u_1, we change the coordinate line, while if we change u_2 we move on the coordinate line. The Gaussian localization implies that the plot of $\kappa_2^{(1)}(u_1, u_2)$ is also localized.

3.2.2. Example: For $N = 3$,

$$\Delta(u_1, u_2, u_3) = 1 + \tfrac{1}{4}[c_{23}^2(\mu_2')^2(\mu_3')^2 + c_{12}^2(\mu_1')^2(\mu_2')^2 + c_{13}^2(\mu_1')^2(\mu_3')^2]$$

and the orthogonal net is

$$\tilde{x}_i = u_i + \frac{1}{\Delta}\left[-\frac{1}{2}\mu_i(c_{ik}^2(\mu_k')^2 + c_{ij}^2(\mu_j')^2) + \sum_{j\neq i}\mu_j\left(c_{ji} + \frac{1}{2}c_{ik}c_{kj}(\mu_k')^2\right)\right]\mu_i'.$$

For the ith coordinate surface we have the following two principal curvatures:

$$\kappa_j^{(i)} = \frac{(c_{ij} + \frac{1}{2}c_{jk}c_{ki}(\mu_k')^2)\mu_i'\mu_j''}{\Delta + [-\frac{1}{2}\mu_i(c_{jk}^2(\mu_k')^2 + c_{ji}^2(\mu_i')^2) + \sum_{l\neq j}\mu_l(c_{lj} + \frac{1}{2}c_{jk}c_{kl}(\mu_k')^2)]\mu_j''} \qquad i \neq j.$$

It is also interesting to consider the first and second fundamental forms of the ith coordinate surface, $\mathrm{I}^{(i)}$ and $\mathrm{II}^{(i)}$, given by

$$\mathrm{I}^{(i)} = H_j^2\,\mathrm{d}u_j^2 + H_k^2\,\mathrm{d}u_k^2$$
$$\mathrm{II}^{(i)} = -\beta_{ij}H_j\,\mathrm{d}u_j^2 - \beta_{ik}H_k\,\mathrm{d}u_k^2$$

with i, j and k cyclic.

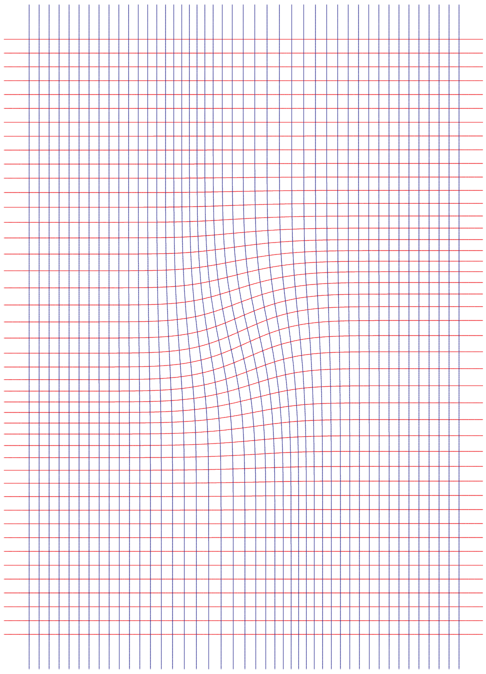


Figure 2.

Elliptic nets. We take as before $\mu(u) = \frac{1}{2}\mathrm{sn}(u|\frac{1}{2})$ and $C = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$. The net is given by

$$\begin{aligned} x_1(u_1,u_2,u_3) = {} & u_1 + 2\mathrm{cn}\,u_1\,\mathrm{dn}\,u_1(-\mathrm{sn}u_2\,(8+\mathrm{cn}^2u_3\mathrm{dn}^2u_3) + \mathrm{sn}u_3\,(-8+\mathrm{cn}^2u_2\,\mathrm{dn}^2u_2) \\ & -(\mathrm{cn}^2u_3\,\mathrm{dn}^2u_3 + \mathrm{cn}^2u_2\,\mathrm{dn}^2u_2)\mathrm{sn}u_1)\{64+\mathrm{cn}^2u_1\,\mathrm{dn}^2u_1\,\mathrm{cn}^2u_2\,\mathrm{dn}^2u_2 \\ & +\mathrm{cn}^2u_1\,\mathrm{dn}^2u_1\,\mathrm{cn}^2u_3\,\mathrm{dn}^2u_3 + \mathrm{cn}^2u_3\,\mathrm{dn}^2u_3\,\mathrm{cn}^2u_2\,\mathrm{dn}^2u_2\}^{-1} \\ x_2(u_1,u_2,u_3) = {} & u_2 + 2\mathrm{cn}\,u_2\,\mathrm{dn}\,u_2(\mathrm{sn}u_1\,(8-\mathrm{cn}^2u_3\mathrm{dn}^2u_3) - \mathrm{sn}u_3\,(8+\mathrm{cn}^2u_1\,\mathrm{dn}^2u_1) \\ & -(\mathrm{cn}^2u_3\,\mathrm{dn}^2u_3 + \mathrm{cn}^2u_1\,\mathrm{dn}^2u_1)\mathrm{sn}u_2)\{64+\mathrm{cn}^2u_1\,\mathrm{dn}^2u_1\,\mathrm{cn}^2u_2\,\mathrm{dn}^2u_2 \\ & +\mathrm{cn}^2u_1\,\mathrm{dn}^2u_1\,\mathrm{cn}^2u_3\,\mathrm{dn}^2u_3 + \mathrm{cn}^2u_3\,\mathrm{dn}^2u_3\,\mathrm{cn}^2u_2\,\mathrm{dn}^2u_2\}^{-1} \\ x_3(u_1,u_2,u_3) = {} & u_3 + 2\mathrm{cn}\,u_3\,\mathrm{dn}\,u_3(\mathrm{sn}u_1\,(8+\mathrm{cn}^2u_2\mathrm{dn}^2u_2) + \mathrm{sn}\,u_2\,(8-\mathrm{cn}^2u_2\,\mathrm{dn}^2u_2) \\ & -(\mathrm{cn}^2u_1\,\mathrm{dn}^2u_1 + \mathrm{cn}^2u_2\,\mathrm{dn}^2u_2)\mathrm{sn}\,u_3)\{64+\mathrm{cn}^2u_1\,\mathrm{dn}^2u_1\,\mathrm{cn}^2u_2\,\mathrm{dn}^2u_2 \\ & +\mathrm{cn}^2u_1\,\mathrm{dn}^2u_1\mathrm{cn}^2u_3\,\mathrm{dn}^2u_3 + \mathrm{cn}^2u_3\,\mathrm{dn}^2u_3\mathrm{cn}^2u_2\,\mathrm{dn}^2u_2\}^{-1}. \end{aligned}$$

Figures 3(*a*) and (*b*) show the surface $u_3 = 0$ and a system of three orthogonal surfaces.

3.3. Dressing Egorov nets

The Egorov reduction is a particular case of the orthogonal reduction and the appropriate dressing kernels $R(z, z')$ must verify (51). Moreover, the Egorov reduction is preserved under the dressing if

$$v \in \tilde{W}^* \Rightarrow v^{\mathrm{t}}(-z) \in \tilde{W}.$$

On the other hand, for $v \in \tilde{W}^*$ we have

$$\frac{\partial v}{\partial \bar{z}} = -\int_A R(z, z')v(z')\,\mathrm{d}^2 z'$$

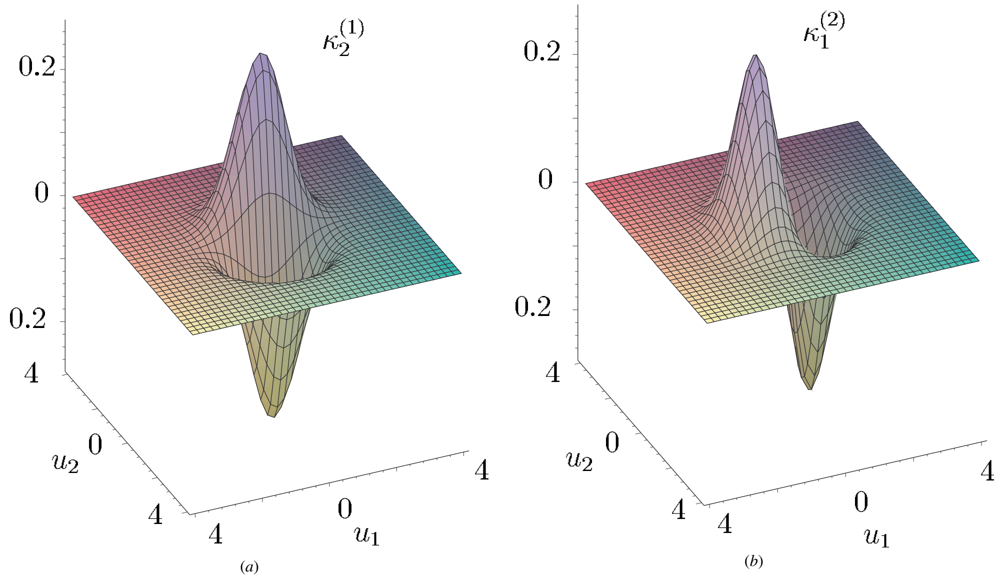


Figure 3.

and thus

$$\frac{\partial(v^{\mathrm{t}}(-z))}{\partial\bar{z}} = \int_A (v^{\mathrm{t}}(-z'))R^{\mathrm{t}}(-z,-z')\,\mathrm{d}^2z'.$$

Thus the dressing kernel must satisfy

$$\begin{cases} zR(z,z') = z'R(-z',-z)^{\mathrm{t}} \\ R(z,z') = R(-z',-z)^{\mathrm{t}}. \end{cases}$$

Hence, $(z-z')R(z,z')=0$ and we are led to the expression

$$R(z,z') = R_0(z)\delta(z-z') \qquad R_0(z) = R_0(-z)^{\mathrm{t}}.$$

Examples of separable kernels of this type are

$$R_0(z) = \pi\sum_{k=1}^{n}[C_k\delta(z-p_k) + C_k^{\mathrm{t}}\delta(z+p_k)]$$

where C_k are $N\times N$ complex matrices and $p_k\in\mathbb{C}$, i.e.

$$R(z,z') = \pi\sum_{k=1}^{n}[C_k\delta(z-p_k)\delta(z'-p_k) + C_k^{\mathrm{t}}\delta(z+p_k)\delta(z'+p_k)]. \tag{53}$$

The separable kernel in the Egorov case induces a dressing transformation with, in principle, singularity problems. These can be fixed if one retraces its source. We first replace our Egorov kernel by

$$R(z,z') = \pi\sum_{k=1}^{n}[C_k\delta(z-p_k)\delta(z'-q_k) + C_k^{\mathrm{t}}\delta(z+p_k)\delta(z'+q_k)]$$

with $q_k\neq p_k$ and then we take the limit $q_k\to p_k$. According to the scheme presented in [14], in order to obtain the dressed net it is required to solve the following matrix equation:

$$\boldsymbol{\lambda}(\boldsymbol{C}-\boldsymbol{C}\boldsymbol{\omega}\boldsymbol{C}) = \boldsymbol{\mu}\boldsymbol{C}\boldsymbol{\omega}\boldsymbol{C}$$

for $\boldsymbol{\lambda}$. Here $\boldsymbol{C} = \operatorname{diag}(C_1, \dots, C_n, C_1^{\mathrm{t}}, \dots, C_n^{\mathrm{t}})$ and

$$\boldsymbol{\omega} = \begin{pmatrix} \Psi(p_1, q_1) & \cdots & \Psi(p_n, q_1) & \Psi(-p_1, q_1) & \cdots & \Psi(-p_n, q_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ \Psi(p_1, q_n) & \cdots & \Psi(p_n, q_n) & \Psi(-p_1, q_n) & \cdots & \Psi(-p_n, q_n) \\ \Psi(p_1, -q_1) & \cdots & \Psi(p_n, -q_1) & \Psi(-p_1, -q_1) & \cdots & \Psi(-p_n, -q_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ \Psi(p_1, -q_n) & \cdots & \Psi(p_n, -q_1) & \Psi(-p_1, -q_n) & \cdots & \Psi(-p_n, -q_n) \end{pmatrix}.$$

From the Egorov reduction, we have that the Cauchy propagator is expressed in terms of Baker functions as

$$\Psi(z, z') = \frac{\psi(-z')^{\mathrm{t}}\psi(z)}{z - z'}.$$

Hence, it is clear that we have a singular behaviour as $q_k \to p_k$. However, observe that all the possible singularities appear in $C_k \Psi(p_k, q_k) C_k$ and in $C_k^t \Psi(-p_k, -q_k) C_k^{\mathrm{t}}$ when we let $q_k \to p_k$. Thus, we can take advantage of the presence of the matrix C_k to cancel these singularities. As $z \to z'$ the Cauchy propagator behaves as

$$\Psi(z, z') = \frac{1}{z - z'} + \psi(-z)^{\mathrm{t}} \frac{\mathrm{d}\psi}{\mathrm{d}z}(z) + \mathcal{O}(z - z').$$

Then, if we take

$$C_k^2 = 0 \qquad k = 1, \dots, n$$

we have

$$\lim_{q_k \to p_k} C_k \Psi(p_k, q_k) C_k = C_k \psi(-p_k)^{\mathrm{t}} \frac{\mathrm{d}\psi}{\mathrm{d}z}(p_k) C_k$$

and there is no singularity at all. We can replace $\boldsymbol{\omega}$ by an effective matrix $\boldsymbol{\omega}_{\mathrm{e}}$ with no singularities of the form

$$\boldsymbol{\omega}_{\mathrm{e}} = \begin{pmatrix} \boldsymbol{\Omega} & -\boldsymbol{\Omega}_- \\ \boldsymbol{\Omega}_+ & \boldsymbol{\Omega}^{\mathrm{t}} \end{pmatrix}$$

with

$$\boldsymbol{\Omega} := \begin{pmatrix} \psi(-p_1)^{\mathrm{t}}\psi'(p_1) & \frac{\psi(-p_1)^{\mathrm{t}}\psi(p_2)}{p_2 - p_1} & \cdots & \frac{\psi(-p_1)^{\mathrm{t}}\psi(p_n)}{p_n - p_1} \\ \frac{\psi(-p_2)^{\mathrm{t}}\psi(p_1)}{p_1 - p_2} & \psi(-p_2)^{\mathrm{t}}\psi'(p_2) & \cdots & \frac{\psi(-p_1)^{\mathrm{t}}\psi(p_n)}{p_n - p_1} \\ \vdots & \vdots & & \vdots \\ \frac{\psi(-p_n)^{\mathrm{t}}\psi(p_1)}{p_1 - p_n} & \frac{\psi(-p_n)^{\mathrm{t}}\psi(p_2)}{p_2 - p_n} & \cdots & \psi(-p_n)^{\mathrm{t}}\psi'(p_n) \end{pmatrix}$$

$$\boldsymbol{\Omega}_\pm := \begin{pmatrix} \frac{\psi(\pm p_1)^{\mathrm{t}}\psi(\pm p_1)}{2p_1} & \cdots & \frac{\psi(\pm p_1)^{\mathrm{t}}\psi(\pm p_n)}{p_1 + p_n} \\ \vdots & & \vdots \\ \frac{\psi(\pm p_n)^{\mathrm{t}}\psi(\pm p_1)}{p_1 + p_n} & \cdots & \frac{\psi(\pm p_n)^{\mathrm{t}}\psi(\pm p_n)}{2p_n} \end{pmatrix}.$$

For $\boldsymbol{\mu}(z)$ and $\boldsymbol{\nu}(z)$ we have

$$\boldsymbol{\mu}(z) = \psi(-z)^{\mathrm{t}}(\boldsymbol{\mu}_+(z), \boldsymbol{\mu}_-(z)) \qquad \boldsymbol{\nu}(z) := \begin{pmatrix} \boldsymbol{\mu}_-^{\mathrm{t}}(-z) \\ \boldsymbol{\mu}_+^{\mathrm{t}}(-z) \end{pmatrix} \psi(z)$$

$$\boldsymbol{\mu}_\pm(z) := \left(\frac{\psi(\pm p_1)}{\pm p_1 - z}, \dots, \frac{\psi(\pm p_n)}{\pm p_n - z} \right).$$

Then, if we denote $\mathfrak{C} = \mathrm{diag}(C_1, \dots, C_n)$ the dressed Cauchy propagator can be written as

$$\tilde{\Psi}(z, z') : = \Psi(z, z') + \psi(-z')^{\mathrm{t}}(\boldsymbol{\mu}_+(z')\mathfrak{C}, \boldsymbol{\mu}_-(z')\mathfrak{C}^{\mathrm{t}}) \begin{pmatrix} 1 - \boldsymbol{\Omega}\mathfrak{C} & \boldsymbol{\Omega}_-\mathfrak{C}^{\mathrm{t}} \\ -\boldsymbol{\Omega}_+\mathfrak{C} & 1 - \boldsymbol{\Omega}^{\mathrm{t}}\mathfrak{C}^{\mathrm{t}} \end{pmatrix}^{-1} \times \begin{pmatrix} \boldsymbol{\mu}_-^{\mathrm{t}}(-z) \\ \boldsymbol{\mu}_+^{\mathrm{t}}(-z) \end{pmatrix} \psi(z).$$

If we assume that $1 - \boldsymbol{\Omega}\mathfrak{C}$ is invertible, then

$$\begin{pmatrix} 1 - \boldsymbol{\Omega}\mathfrak{C} & \boldsymbol{\Omega}_-\mathfrak{C}^{\mathrm{t}} \\ -\boldsymbol{\Omega}_+\mathfrak{C} & 1 - \boldsymbol{\Omega}^{\mathrm{t}}\mathfrak{C}^{\mathrm{t}} \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ \boldsymbol{\gamma} & \boldsymbol{\delta} \end{pmatrix}$$

where

$$\begin{aligned}
\boldsymbol{\alpha} &:= [1 - \boldsymbol{\Omega}\mathfrak{C} + \boldsymbol{\Omega}_-\mathfrak{C}^{\mathrm{t}}(1 - \boldsymbol{\Omega}^{\mathrm{t}}\mathfrak{C}^{\mathrm{t}})^{-1}\boldsymbol{\Omega}_+\mathfrak{C}]^{-1} \\
\boldsymbol{\beta} &:= -[1 - \boldsymbol{\Omega}\mathfrak{C} + \boldsymbol{\Omega}_-\mathfrak{C}^{\mathrm{t}}(1 - \boldsymbol{\Omega}^{\mathrm{t}}\mathfrak{C}^{\mathrm{t}})^{-1}\boldsymbol{\Omega}_+\mathfrak{C}]^{-1}\boldsymbol{\Omega}_-\mathfrak{C}^{\mathrm{t}}(1 - \boldsymbol{\Omega}^{\mathrm{t}}\mathfrak{C}^{\mathrm{t}})^{-1} \\
\boldsymbol{\gamma} &:= [1 - \boldsymbol{\Omega}^{\mathrm{t}}\mathfrak{C}^{\mathrm{t}} + \boldsymbol{\Omega}_+\mathfrak{C}(1 - \boldsymbol{\Omega}\mathfrak{C})^{-1}\boldsymbol{\Omega}_-\mathfrak{C}^{\mathrm{t}}]^{-1}\boldsymbol{\Omega}_+\mathfrak{C}(1 - \boldsymbol{\Omega}\mathfrak{C})^{-1} \\
\boldsymbol{\delta} &:= [1 - \boldsymbol{\Omega}^{\mathrm{t}}\mathfrak{C}^{\mathrm{t}} + \boldsymbol{\Omega}_+\mathfrak{C}(1 - \boldsymbol{\Omega}\mathfrak{C})^{-1}\boldsymbol{\Omega}_-\mathfrak{C}^{\mathrm{t}}]^{-1}
\end{aligned}$$

so that we finally get the following expression for the dressed Cauchy propagator:

$$\begin{aligned}
\tilde{\Psi}(z, z') = \Psi(z, z') + \psi(-z')^{\mathrm{t}}[&\boldsymbol{\mu}_+(z')\mathfrak{C}\boldsymbol{\alpha}\boldsymbol{\mu}_-^{\mathrm{t}}(-z) + \boldsymbol{\mu}_+(z')\mathfrak{C}\boldsymbol{\beta}\boldsymbol{\mu}_+^{\mathrm{t}}(-z) + \boldsymbol{\mu}_-(z')\mathfrak{C}^{\mathrm{t}}\boldsymbol{\gamma}\boldsymbol{\mu}_-^{\mathrm{t}}(-z) \\
&+\boldsymbol{\mu}_-(z')\mathfrak{C}^{\mathrm{t}}\boldsymbol{\delta}\boldsymbol{\mu}_+^{\mathrm{t}}(z)]\psi(z).
\end{aligned}$$

3.3.1. Elementary dressing of the vacuum. The simplest dressing of the vacuum

$$\psi(z, \boldsymbol{t}) = \exp(\xi(z, \boldsymbol{t})) \qquad \psi(-z)^{\mathrm{t}}\psi'(z) = \xi'(z, \boldsymbol{t})$$

corresponds to $R_0(z) = C\delta(z - p) + C^{\mathrm{t}}\delta(z + p)$, where $C = \sum_{i\in I, j\in I'} c_{ij}E_{ij}$ with $I, I' \subset \{1, \dots, N\}$ and $I \cap I' = \emptyset$. Examples of these types of matrices are $\sum_{i\neq j} c_i E_{ij}$. In this case the dressing simplifies and we have

$$\boldsymbol{\mu}_\pm(z) = \frac{\mathrm{e}^{\pm\xi(p)}}{\pm p - z} \qquad \boldsymbol{\Omega} = \frac{\partial\xi}{\partial z}(p), \qquad \boldsymbol{\Omega}_\pm := \frac{\mathrm{e}^{\pm 2\xi(p)}}{2p}$$

$$\boldsymbol{\alpha} = \left[1 + \frac{1}{4p^2}\mathrm{e}^{-2\xi(p)}C^{\mathrm{t}}\mathrm{e}^{2\xi(p)}C\right]^{-1} \qquad \boldsymbol{\beta} = -\left[1 + \frac{1}{4p^2}\mathrm{e}^{-2\xi(p)}C^{\mathrm{t}}\mathrm{e}^{2\xi(p)}C\right]^{-1}\frac{\mathrm{e}^{-2\xi(p)}}{2p}C^{\mathrm{t}}$$

$$\boldsymbol{\gamma} = \left[1 + \frac{1}{4p^2}\mathrm{e}^{2\xi(p)}C\mathrm{e}^{-2\xi(p)}C^{\mathrm{t}}\right]^{-1}\frac{\mathrm{e}^{2\xi(p)}}{2p}C \qquad \boldsymbol{\delta} = \left[1 + \frac{1}{4p^2}\mathrm{e}^{2\xi(p)}C\mathrm{e}^{-2\xi(p)}C^{\mathrm{t}}\right]^{-1}$$

where $\xi(p) := \xi(p, \mathrm{u})$. We have also taken into account that, because of the diagonal character of $\frac{\partial\xi}{\partial z}$, we get $C\frac{\partial\xi}{\partial z}C = 0$, so that it can be deleted from the effective matrix $\boldsymbol{\omega}_{\mathrm{e}}$. Thus, the Cauchy propagator is

$$\begin{aligned}
\tilde{\Psi}(z, z') = \mathrm{e}^{-\xi(z')}\bigg[&\frac{1}{z - z'} + \frac{\mathrm{e}^{\xi(p)}}{p - z'}C\left[1 + \frac{1}{4p^2}\mathrm{e}^{-2\xi(p)}C^{\mathrm{t}}\mathrm{e}^{2\xi(p)}C\right]^{-1}\frac{\mathrm{e}^{-\xi(p)}}{z - p} \\
&- \frac{\mathrm{e}^{\xi(p)}}{p - z'}C\left[1 + \frac{1}{4p^2}\mathrm{e}^{-2\xi(p)}C^{\mathrm{t}}\mathrm{e}^{2\xi(p)}C\right]^{-1}\frac{\mathrm{e}^{-2\xi(p)}}{2p}C^{\mathrm{t}}\frac{\mathrm{e}^{\xi(p)}}{z + p} \\
&- \frac{\mathrm{e}^{-\xi(p)}}{p + z'}C^{\mathrm{t}}\left[1 + \frac{1}{4p^2}\mathrm{e}^{2\xi(p)}C\mathrm{e}^{-2\xi(p)}C^{\mathrm{t}}\right]^{-1}\frac{\mathrm{e}^{2\xi(p)}}{2p}C\frac{\mathrm{e}^{-\xi(p)}}{z - p} \\
&- \frac{\mathrm{e}^{-\xi(p)}}{p + z'}C^{\mathrm{t}}\left[1 + \frac{1}{4p^2}\mathrm{e}^{2\xi(p)}C\mathrm{e}^{-2\xi(p)}C^{\mathrm{t}}\right]^{-1}\frac{\mathrm{e}^{\xi(p)}}{z + p}\bigg]\mathrm{e}^{\xi(z)}.
\end{aligned}$$

The associated nets have no singularities. To see this we notice that the singularities are the points for which

$$\Delta := \det\left[I_{2N} + \frac{1}{2p}\begin{pmatrix} e^{-\xi(p)} & 0 \\ 0 & e^{\xi(p)} \end{pmatrix}\begin{pmatrix} 0 & -C^{t} \\ C & 0 \end{pmatrix}\right]$$

vanishes and then, by applying the same argument as in the orthogonal $n = 1$ case, one gets $\Delta \geqslant 1$.

It is not difficult to prove that the matrix C can be expressed in the form

$$C = \sum_{\substack{i=1,\dots,r\\ k=r+1,\dots,N}} c_{ik}E_{ik} \qquad 1 \leqslant r < N.$$

By introducing the matrices

$$e = \sum_{i,j=1}^{r} e_{ij}E_{ij} \qquad e_{ij} = \frac{e^{2\xi_i(p)}}{4p^2}\sum_{k=r+1}^{N} c_{ik}c_{jk}e^{-2\xi_k(p)} \qquad \tilde{e} = I_r(I_N + e)^{-1}I_r$$

where $I_r = \sum_{i=1}^{r} E_{ii}$, we can state the following proposition.

Proposition 3. *The function*

$$\tilde{\Psi}(z,z') = e^{-\xi(z')}\left[\frac{1}{z-z'} + \begin{pmatrix} \frac{2p}{(z'-p)(z+p)}(I_r - e^{-\xi(p)}\tilde{e}\,e^{\xi(p)}) & -\frac{1}{(z'-p)(z-p)}e^{-\xi(p)}\tilde{e}\,e^{2\xi(p)}Ce^{-\xi(p)} \\ -\frac{1}{(z'+p)(z+p)}e^{-\xi(p)}C^{t}\tilde{e}\,e^{\xi(p)} & -\frac{1}{2p(z'+p)(z-p)}e^{-\xi(p)}C^{t}\tilde{e}\,e^{2\xi(p)}Ce^{-\xi(p)} \end{pmatrix}\right]e^{\xi(z)}$$

is a Cauchy propagator fulfilling the Egorov reduction.

3.3.2. Example: For $r = 1$ one finds

$$e = \frac{e^{2\xi_1(p)}}{4p^2}\sum_{k=2}^{N} c_k^2 e^{-2\xi_k(p)} \qquad \tilde{e} = \frac{1}{1 + \frac{e^{2\xi_1(p)}}{4p^2}\sum_{k=2}^{N} c_k^2 e^{-2\xi_k(p)}}.$$

and the Cauchy propagator is

$$\tilde{\Psi}(z,z') = \frac{e^{\xi(z)-\xi(z')}}{z-z'} + e^{-\xi(z')}\tilde{e}\begin{pmatrix} \frac{2pe}{(z'-p)(z+p)} & -\frac{c_2e^{\xi_1(p)-\xi_2(p)}}{(z'-p)(z-p)} & \cdots & -\frac{c_Ne^{\xi_1(p)-\xi_N(p)}}{(z'-p)(z-p)} \\ -\frac{c_2e^{\xi_1(p)-\xi_2(p)}}{(z'-p)(z-p)} & -\frac{c_2^2e^{2(\xi_1(p)-\xi_2(p))}}{2p(z'+p)(z-p)} & \cdots & -\frac{c_2c_Ne^{2\xi_1(p)-\xi_2(p)-\xi_N(p)}}{2p(z'+p)(z-p)} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{c_Ne^{\xi_1(p)-\xi_N(p)}}{(z'-p)(z-p)} & -\frac{c_Nc_2e^{2\xi_1(p)-\xi_2(p)-\xi_N(p)}}{2p(z'+p)(z-p)} & \cdots & -\frac{c_N^2e^{2(\xi_1(p)-\xi_N(p))}}{2p(z'+p)(z-p)} \end{pmatrix}e^{\xi(z)}.$$

Observe that the higher-times dependence can be absorbed in the constants c_j. Thus we perform the replacement $\xi_i(p) \mapsto pu_i$. By setting $\mathcal{N} = \mathcal{M} = I_N$ we get the following net:

$$x_1 = u_1 - \frac{1}{2p^3}\frac{\sum_{k=2}^{N} c_k^2 e^{2p(u_1-u_k)}}{1 + \frac{1}{4p^2}\sum_{k=2}^{N} c_k^2 e^{2p(u_1-u_k)}}$$

$$x_j = -\frac{1}{p^2}\frac{c_je^{p(u_1-u_j)}}{1 + \frac{1}{4p^2}\sum_{k=2}^{N} c_k^2 e^{2p(u_1-u_k)}} \qquad j = 2,\dots,N.$$

By noticing that

$$x_2^2+\cdots+x_N^2=\frac{4}{p^2}e\,\tilde{e}^2=\frac{4}{p^2}(\tilde{e}-\tilde{e}^2)$$

one deduces that

$$\tilde{e}=\tfrac{1}{2}\left(1-\sqrt{1-p^2(x_2^2+\cdots+x_N^2)}\right)$$

which allows us to write the inverse map as

$$u_1=x_1+\frac{1}{p}\left[1+\sqrt{1-p^2(x_2^2+\cdots+x_N^2)}\right]$$

$$u_j=x_1+\frac{1}{p}\left[1+\sqrt{1-p^2(x_2^2+\cdots+x_N^2)}\right.$$
$$\left.-\ln\left(-\frac{2x_j\left[1+\sqrt{1-p^2(x_2^2+\cdots+x_N^2)}\right]}{c_j(x_2^2+\cdots+x_N^2)}\right)\right]$$

where $j=2,\ldots,N$. The deformed flat coordinates are

$$\theta_1(z)=\frac{\mathrm{e}^{zu_1}-1}{z}-\frac{1}{2p^2}\frac{\mathrm{e}^{2pu_1}\sum_{k=2}^N c_k^2\mathrm{e}^{-2pu_k}}{1+\frac{1}{4p^2}\sum_{k=2}^N c_k^2\mathrm{e}^{2p(u_1-u_k)}}\frac{\mathrm{e}^{zu_1}}{z+p}$$

$$\theta_j(z)=\frac{1}{p}\frac{c_j\mathrm{e}^{p(u_1-u_j)}}{1+\frac{1}{4p^2}\sum_{k=2}^N c_k^2\mathrm{e}^{2p(u_1-u_k)}}\frac{\mathrm{e}^{zu_j}}{z-p}\qquad j=2,\ldots,N$$

and the densities $h_{j,n-1}$, $n=1,2,\ldots$ are given by

$$h_{1,n-1}=\frac{u_1^{n+1}}{(n+1)!}+\frac{1}{2p^2}\frac{\sum_{k=2}^N c_k^2\mathrm{e}^{2p(u_1-u_k)}}{1+\frac{1}{4p^2}\sum_{k=2}^N c_k^2\mathrm{e}^{2p(u_1-u_k)}}\sum_{k=0}^{n}\frac{u_1^k}{k!(-p)^{n-k+1}}$$

$$h_{j,n-1}=-\frac{1}{p}\frac{c_j\mathrm{e}^{p(u_1-u_j)}}{1+\frac{1}{4p^2}\sum_{k=2}^N c_k^2\mathrm{e}^{2p(u_1-u_k)}}\sum_{k=0}^{n}\frac{u_j^k}{k!p^{n-k+1}}\qquad j=2,\ldots,N.$$

Proposition 4. *The functions*

$$h_{1,n-1}=\frac{\left(x_1+\frac{1}{p}\left[1+\sqrt{1-p^2(x_2^2+\cdots+x_N^2)}\right]\right)^{n+1}}{(n+1)!}+\left[1+\sqrt{1-p^2(x_2^2+\cdots+x_N^2)}\right]$$
$$\times\sum_{k=0}^{n}\frac{\left(x_1+\frac{1}{p}\left[1+\sqrt{1-p^2(x_2^2+\cdots+x_N^2)}\right]\right)^k}{k!(-p)^{n-k+1}}$$

$$h_{j,n-1}=px_j\sum_{k=0}^{n}\frac{\left(x_1+\frac{1}{p}\left[1+\sqrt{1-p^2(x_2^2+\cdots+x_N^2)}-\ln\left(-\frac{2x_j\left[1+\sqrt{1-p^2(x_2^2+\cdots+x_N^2)}\right]}{c_j(x_2^2+\cdots+x_N^2)}\right)\right]\right)^k}{k!p^{n-k+1}}$$

with $j=2,\ldots,N$, are Hamiltonian densities generating a hierarchy of integrable systems of hydrodynamic type.

From the expressions

$$h_{1,0} = \tfrac{1}{2}(x_1^2 + x_2^2 + \cdots + x_N^2)$$

$$h_{j,0} = x_j\left(x_1 + \frac{1}{p}\left[2 + \sqrt{1 - p^2(x_2^2 + \cdots + x_N^2)}\right.\right.$$
$$\left.\left. - \ln\left(-\frac{2x_j\left[1 + \sqrt{1 - p^2(x_2^2 + \cdots + x_N^2)}\right]}{c_j(x_2^2 + \cdots + x_N^2)}\right)\right]\right)$$

with $j = 2, \ldots, N$, and since $h_{j,0} = \frac{\partial F}{\partial x_j}$, we conclude the following proposition.

Proposition 5. *The following function:*

$$F(x_1, \ldots, x_N) = \frac{1}{6}x_1^3 + \frac{1}{p}(x_2^2 + \cdots + x_N^2) + \frac{x_1}{2}(x_2^2 + \cdots + x_N^2)$$
$$+\frac{1}{6p^3}[1 + 2p^2(x_2^2 + \cdots + x_N^2)]\sqrt{1 - p^2(x_2^2 + \cdots + x_N^2)}$$
$$-\sum_{j=2}^{N}\frac{x_j^2}{2p}\ln\left[-\frac{2x_j}{c_j}\frac{1 + \sqrt{1 - p^2(x_2^2 + \cdots + x_N^2)}}{x_2^2 + \cdots + x_N^2}\right]$$

satisfies the WDVV associativity equations (7).

3.3.3. Example: As a final example let us consider the case $N = 2$ and

$$\mathcal{N} = \mathcal{M}^t = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{54}$$

The corresponding net becomes

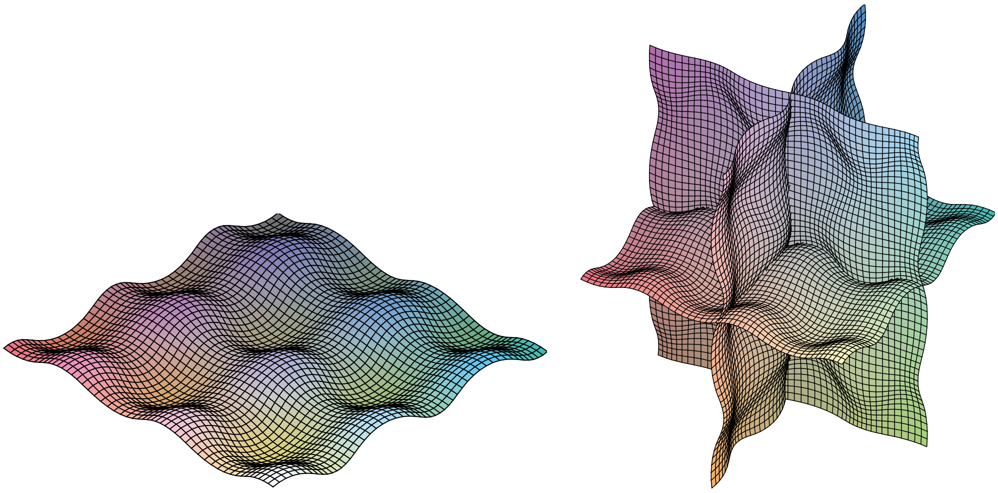
$$\begin{aligned} x_1 &= u_1 + u_2 + \frac{2c\mathrm{e}^{p(u_1-u_2)}}{p^2\left(1 + \frac{1}{4p^2}c^2\mathrm{e}^{2p(u_1-u_2)}\right)} \\ x_2 &= u_1 - u_2 - \frac{c^2\mathrm{e}^{2p(u_1-u_2)}}{p^3\left(1 + \frac{1}{4p^2}c^2\mathrm{e}^{2p(u_1-u_2)}\right)} \end{aligned} \tag{55}$$

with $\Delta = 1 + \frac{1}{4p^2}c^2\mathrm{e}^{-2p(u_1-u_2)}$.

The deformed flat coordinates are now given by

$$\theta_1(z) = \frac{\mathrm{e}^{zu_1} + \mathrm{e}^{zu_2} - 2}{z} + \frac{a\mathrm{e}^{p(u_1-u_2)}(2p - a\mathrm{e}^{p(u_1-u_2)})}{2p^2\left(1 + \frac{1}{4p^2}c^2\mathrm{e}^{2p(u_1-u_2)}\right)}\frac{\mathrm{e}^{zu_1}}{z + p}$$
$$-\frac{a\mathrm{e}^{p(u_1-u_2)}(2p + a\mathrm{e}^{p(u_1-u_2)})}{2p^2\left(1 + \frac{1}{4p^2}c^2\mathrm{e}^{2p(u_1-u_2)}\right)}\frac{\mathrm{e}^{zu_2}}{z - p}$$

$$\theta_2(z) = \frac{\mathrm{e}^{zu_1} - \mathrm{e}^{zu_2}}{z} + \frac{a\mathrm{e}^{p(u_1-u_2)}(2p - a\mathrm{e}^{p(u_1-u_2)})}{2p^2\left(1 + \frac{1}{4p^2}c^2\mathrm{e}^{2p(u_1-u_2)}\right)}\frac{\mathrm{e}^{zu_1}}{z + p}$$
$$+\frac{a\mathrm{e}^{p(u_1-u_2)}(2p + a\mathrm{e}^{p(u_1-u_2)})}{2p^2\left(1 + \frac{1}{4p^2}c^2\mathrm{e}^{2p(u_1-u_2)}\right)}\frac{\mathrm{e}^{zu_2}}{z - p}.$$



(*a*) (*b*)

Figure 4.

From (9) one finds that the expression of the free energy function in terms of the coordinates (u_1, u_2) is

$$\begin{aligned}F(x_1, x_2) = {} & \frac{4}{p^2}(u_1+u_2) - \frac{2}{p}(u_1^2-u_2^2) + \frac{1}{3}(u_1^3+u_2^3) \\ & + \frac{4a}{p^4\left(1+\frac{1}{4p^2}c^2\mathrm{e}^{2p(u_1-u_2)}\right)^2}[p(u_1-u_2)-2]\mathrm{e}^{p(u_1-u_2)} \\ & + \frac{1}{p^4\left(1+\frac{1}{4p^2}c^2\mathrm{e}^{2p(u_1-u_2)}\right)}[(6-4p(u_1-u_2)+p^2(u_1^2+u_2^2))a\mathrm{e}^{p(u_1-u_2)} \\ & -4p^2(u_1+u_2)+2p^3(u_1^2-u_2^2)].\end{aligned}$$

There are three sectors in $\mathbb{R}^2$ where (55) is an invertible map. For $a,\ p \geqslant 0$, regularity is absent on the lines

$$u_1 - u_2 = w_\pm := \frac{1}{p}\ln\left(\frac{2p}{c}(\sqrt{2}\pm 1)\right)$$

(where the Jacobian of the transformation $(u_1, u_2) \mapsto (x_1, x_2)$ vanishes). Thus there are two charts:

$$\mathcal{U}_\pm := \{(u_1, u_2) \in \mathbb{R}^2 : u_1 - u_2 \gtrless w_\pm\}$$

as shown in figures 4(*a*) and (*b*), which map into the regions

$$R_\pm = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \gtrless f(w_\pm)\}$$

where

$$f(w) := w - \frac{c^2\mathrm{e}^{2pw}}{p^3\left(1+\frac{1}{4p^2}c^2\mathrm{e}^{2pw}\right)}.$$

Hence, this net determines an atlas with two charts. We now plot the coordinate lines in each of these charts in figure 5.

The plot of both systems of coordinate lines is given in figure 6.

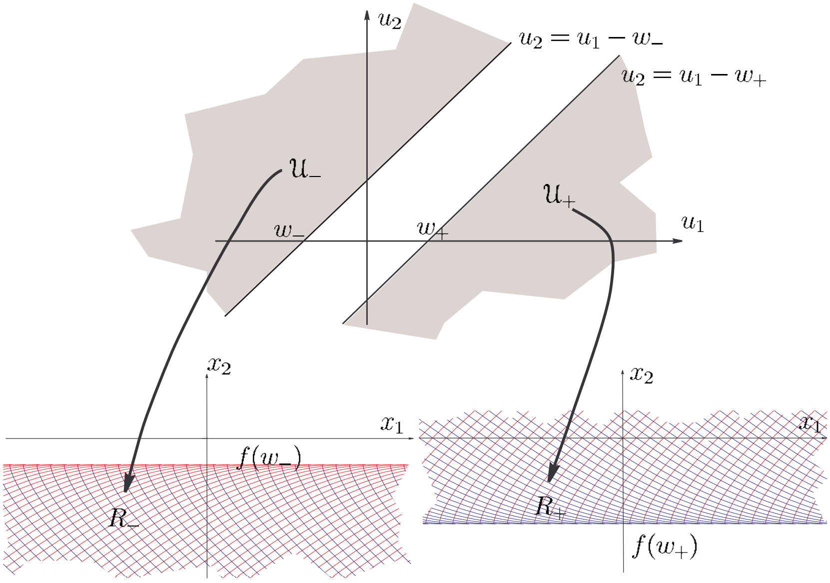


Figure 5.

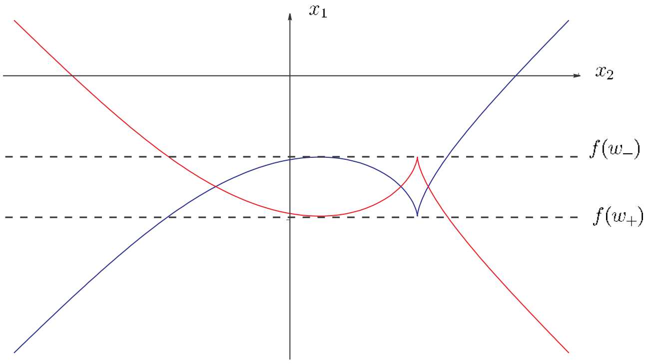


Figure 6.

For the $u_1 = $ constant coordinate lines, we have $x_1 + x_2 \sim 0$ for $u_2 \to +\infty$, while $x_1 + x_2 + \frac{4}{p} = 0$ for $u_2 \to -\infty$. On the other hand, for the $u_2 =$ constant coordinate lines, $x_1 - x_2 + \frac{4}{p} \sim 0$ for $u_1 \to +\infty$ and $x_1 - x_2 = 0$ for $u_1 \to -\infty$. Thus, far away from the $x_2 = 0$ line the coordinate lines form a Cartesian net. However, there is exponentially localized deformation on the region bounded by the lines $x_2 = f(w_\pm)$, the overlapping of the two patches used in the net. Moreover, when the coordinate lines goes trough this region they acquire a shift equal to $\frac{4}{p}$. This property is reminiscent of the solitonic character of the integrable systems.

4. Concluding remarks and outlook

Self-similar solutions of the WDVV are particularly relevant in the classification problem of topological conformal field theories. In the decomposable case these solutions correspond to

∂-invariant Egorov nets with rotation coefficients satisfying [6–10]

$$\sum_{i=1}^{n} u_i \frac{\partial \beta_{lk}}{\partial u_i} = -\beta_{lk} \tag{56}$$

or equivalently $\beta(cu) = c^{-1}\beta(u)$, with c being a nonvanishing constant. From (13) and (16), we deduce that (56) holds, provided the Baker function fulfils the condition

$$\psi(cz, \mathrm{u}) = \psi(z, \mathrm{u}(c))$$

where $u(c)_{in} := c^n u_{in}$, $i = 1, 2, \ldots, N$, $n \in \mathbb{N}$. This condition is preserved under a dressing transformation if the kernel $R(z', z)$ satisfies

$$R(cz', cz) = c^{-3} R(z', z).$$

In particular, for kernels of Egorov type this condition is

$$R_0(cz) = c^{-1} R_0(z). \tag{57}$$

We notice that the examples analysed above do not satisfy (57). Indeed, (57) does not hold if $R_0(z)$ has a bounded support, which is just the class of kernels used in our scheme. This means that the solution methods of this paper require an appropriate generalization in order to make them suitable to generate self-similar solutions of the WDVV equations.

Dispersionless limits of integrable systems lead to integrable systems of hydrodynamic type [7, 15]. An interesting problem worth considering is to apply the KP formalism to investigate the existence of integrable models (reductions of multi-component KP hierarchies) the dispersionless limit of which are the hydrodynamic systems provided by the ∂-invariant Egorov nets. According to [7–10] these integrable models, in the self-similar case, would allow us to reconstruct the corresponding underlying topological field theory.

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References

[1] Darboux G 1910 *Leçons Sur le Systèmes Orthogonaux et Les Coordonnées Curvilignes* (Paris)
[2] Doliwa A, Manakov S V and Santini P M 1998 *Commun. Math. Phys.* **196** 1
Doliwa A and Santini P M *Preprint solv-int/9907012*
[3] Doliwa A, Santini P M and Mañas M 2000 *J. Math. Phys.* **41** 944
[4] Dubrovin B and Novikov S P 1983 *Dokl. Akad. Nauk* **27** 665
[5] Tsarev S P 1985 *Sov. Math.–Dokl.* **31** 488
[6] Dubrovin B 1992 *Nucl. Phys.* B **379** 627
[7] Dubrovin B 1994 Geometry of 2D topological field theory *Preprint* SISSA-89/94/FM
Dubrovin B 1994 *Preprint* hep-th/9407018
[8] Dubrovin B and Novikov S P 1989 *Usp. Mat. Nauk* **44** 35
[9] Dubrovin B and Zhang Y 1998 *Commun. Math. Phys.* **198** 311
[10] Dubrovin B and Zhang Y 1998 Frobenius manifolds and Virasoro constraints *Preprint* math.AG/9808048
[11] Dijkgraaf R, Verlinde E and Verlinde H 1991 *Nucl. Phys.* B **325** 59
[12] Krichever I M 1997 *Funct. Anal. Appl.* **31** 25
[13] Lamé G 1859 *Leçons Sur la Théorie des Coordonnées Curvilignes et Leurs Diverses Applications* (Paris)
[14] Mañas M, Alonso L M and Medina E 2000 *J. Phys. A: Math. Gen.* **33** 2871
[15] Takasaki K and Takebe T 1992 *Int. J. Mod. Phys.* A (Suppl 1B) **7** 889
[16] van der Leur J W and Martini R 1999 *Commun. Math. Phys.* **205** 587
van der Leur J W 2000 *Twisted GL_n Loop Group Orbit and Solutions of the WDVV Equations* vol 2 arXiv:nlin.SI/0004021
[17] Witten E 1990 *Nucl. Phys.* B **340** 281
[18] Zakharov V E 1998 *Duke Math. J.* **94** 103